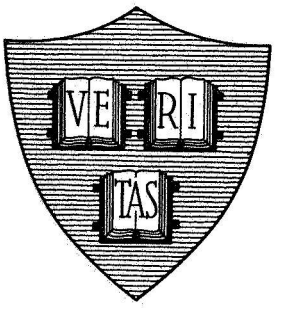
ANALYSIS AND DESIGN OF INTEGRATION FORMULAS FOR A RANDOM INTEGRAND

Technical Report No. 2



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

In this report, an analysis of known integration formulas under random circumstances are presented and a procedure for designing an "optimal" integration formula which takes advantage of the statistical knowledge of the integrand is developed. The optimal integration formula of a different degree is chosen according to the capacity of the onboard computer and the required output frequency of the integrated value of the integrand. It is shown that a higher degree optimal integration formula which carries less of a computational load can be effectively derived from the second degree optimal integration formula. This will reduce the computational complexity necessary for obtaining the optimal coefficients of a higher degree integration formula on an off-line computer. To overcome the difficulty of implementing the time varying coefficients of an integration formula, a method is developed for designing the constant coefficient suboptimal integration formula. Computational results obtained by applying these integration formulas show significant improvement over those obtained by using conventional integration rules, e. g., the trapezoid rule, Simpson's rule, or Newton's three-eighths rule.

1. INTRODUCTION

Most of numerical analysis texts [1] [2] deal with numerical integration problems by considering the integrand deterministically known at equally-spaced sampling points. Integration formulas, such as the trapezoid rule, Simpson's rule and Newton's three-eights rule etc., are derived by requiring the integration to be exact for the case when the integrands are polynomials of various order.

There seems to be a scarcity of literature in numerical analysis dealing with the problem when the integrand is a stochastic function or a random process. Such problems arise frequently in real time applications. For example, in an inertial navigation system, the signal from the gyro or accelerometer measurement device is by no means deterministic, but rather a multi-dimensional random process with known statistical properties and, in general, the signal is corrupted by noise. The purpose of this paper is to present a systematic procedure for the development of integration formulas which explicitly takes the statistical property of the integrand into account and which is optimal in the sense of minimizing the accumulated mean square error over the whole integration interval.

For illustration, let us take a simple example.

Let

$$\dot{x} = y(t) \qquad ; x(t_o) \text{ given}$$

where y(t), the integrand, is a purely random process with zero mean and unit variance. If the observations of the integrand at the sampling points are $z(t_i) = y(t_i)$; $i = 0, 1, \ldots, N$, then it is intuitively reasonable to assert that the optimal integration formula is simple $\hat{x}(t_{i+1}) = \hat{x}(t_i) + dz(t_i)$; $x(t_o) = \hat{x}(t_o)$, where d is the sampling step size and $\hat{x}(t_i)$ denotes the

computed value of $x(t_i)$. Any other formula, e. g. Simpson's rule with $x^*(t_{i+2}) = x^*(t_i) + \frac{d}{3}[z(t_i) + 4z(t_{i+1}) + z(t_{i+2})]$, will yield a larger mean square error as can be easily verified (see section 3).

In this report, we restrict the consideration to cases where the integrand is a linear function of a Markov random process (see section 2). An analysis and comparison of the mean square integration error of various known integration formulas applied to the above random process are given in section 3. In section 4, the optimal integration formulas are derived by minimizing the cumulated mean square error and are proved to be related to the Kalman-Bucy estimator which is well-known in control theory [4]. However, the computational load of the optimal integration formula increases with the complexity of the random process. A procedure for designing an optimal integration formula subject to computational complexity constraints is presented in section 5. The results of applying the derived formula to typical examples are compared with most commonly used integration formulas (e. g. the trapezoid rule and Simpson's rule).

The types of complexity constraints considered are primarily due to the limit on fast memory or storage. We consider generally the class of optimal integration formulas which has the form

$$\hat{x}(i + \ell - 1) = \hat{x}(i) + \sum_{j=1}^{\ell} a_j(i) z(i + j - 1)$$

where $z(\cdot)$ is the measured value of the integrand at various time instants and $\hat{x}(\cdot)$ is the calculated value of the integral, $\int y\,dt$. Complexity constraints are imposed by requiring ℓ to be fixed and/or $a_j(i)$ to be piece-wise constant. Finally, the interplay between computational load and output frequency is discussed in section 6. Examples of applications showing the advantages of this approach are given.

2. FORMULATION

In the case that the integrand is a linear function of a Markov random process which can be completely specified by the solution of a set of linear differential equations driven by white noise, that is,

$$\dot{y} = Fy + Gu(t) + rw(t), \tag{2-1}$$

where y(t) is a column vector, w(t) is a purely random process with

$$E\,[w(t)] = 0 \tag{2-2}$$

$$E\,[w(t)w(\tau)^T] = q\delta(t - \tau)$$

and

$$E\,[y(t_o)] = y_o \tag{2-3}$$

$$E\,[y(t_o)y(t_o)^T] = p_o$$

$E\,[\quad]$ = expected value operator.

u(t) is a deterministically known input function. Knowing (2-1), (2-2), and (2-3), is equivalent to specifying the joint probability density of y(t) for any finite collection of time. Then, the integration problem can be formulated by computing the value of $x(t_i)$ for $i = 1, 2, 3, \ldots$ where

$$\dot{x} = h^T y(t); \quad t_o < t < t_N; \quad x(o) = \text{given} \tag{2-4}$$

and measurements on the integrand, $h^T y$, at sampling points, are

$$z(t_i) = h^T y(t_i) + v(t_i) \quad t_i = t_o, t_1, \ldots, t_N \tag{2-5}$$

where h is a vector with the dimension of y(t), v(t) is the measurement noise and is assumed to be a purely random sequence with

$$E\,[v(t_i)] = 0 \tag{2-6}$$

$$E\,[v(t_i)v(t_j)] = r\delta_{ij}$$

In general, the vector h is a constant vector and without losing generality, we assume $h^T = [1, 0, 0]$. For the convenience of later

developments, (2-1) and (2-4) will be discretized at sampling points and to simplify notation, from now on, we shall denote "t_i" by "i", e. g., $z(t_i) = z(i), x(t_i) = x(i)$ and $y(t_i)$. The discrete version of (2-1) through (2-5) is

$$\begin{bmatrix} x(i+1) \\ y(i+1) \end{bmatrix} = \begin{bmatrix} & \\ \Psi(i+1,i) & \\ & \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} + \begin{bmatrix} u_1(i) \\ u_2(i) \end{bmatrix} + \begin{bmatrix} w_1(i) \\ w_2(i) \end{bmatrix} \qquad (2\text{-}7)$$

where

$$\Psi(i+1,i) = \begin{bmatrix} \Psi_{11}, \Psi_{12} \\ \Psi_{21}, \Psi_{22} \end{bmatrix}; \quad \Psi_{11} = 1, \Psi_{21} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1(i) \\ u_2(i) \end{bmatrix} \equiv \int_{t_i}^{t_i+1} \Psi(t_i+1,\tau) \begin{bmatrix} 0 \\ G(\tau) \end{bmatrix} u(\tau)d\tau$$

$$E \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \equiv 0$$

$$E \begin{bmatrix} w_1(i) \\ w_2(i) \end{bmatrix} [w_1(i)^T, w_2(i)^T] = \int_{t_i}^{t_i+1} \Psi(t_{i+1},\tau) q \Phi^T(t_{i+1},\tau) d\tau \triangleq Q^1(i).$$

The measurement (5) is

$$z(i) = h^T y(i) + v(i), \qquad \forall_i. \qquad (2\text{-}8)$$

$\Psi(t_{i+1},\tau)$ is the transition matrix of (2-1) and (2-4). Instead of treating (2-1), (2-4) and (2-5), (2-7) and (2-8) will take their place from now on. It is to be noted that this kind of discretization does not, in principle, introduce discretization errors, since we can precalculate $\Psi(i+1,i), u_1(i), u_2(i)$ and $Q(i)$ to be as accurate as we wish.

The problem then is to design a formula of the type

$$\hat{x}(i+1) = f(x(0), z(0), z(1), z(2), \ldots, z(i+1)) \qquad (2\text{-}10)$$

such that $\hat{x}(i+1)$ is, in some sense, a good approximation to the value $x(i+1)$ of (2-4) or (2-7). If f is given, we have an analysis problem which is discussed next. If f is to be determined, than it is the synthesis problem, to be detailed in sections 4 to 6.

3. ANALYSIS OF KNOWN INTEGRATION FORMULAS

The error terms produced by applying integration rules on a deterministic integrand are expressed in terms of the higher order derivative of the integrand as they are listed in numerical analysis textbooks. In our case, where y(t) are random processes, there must be random errors generated in addition to the deterministic errors.

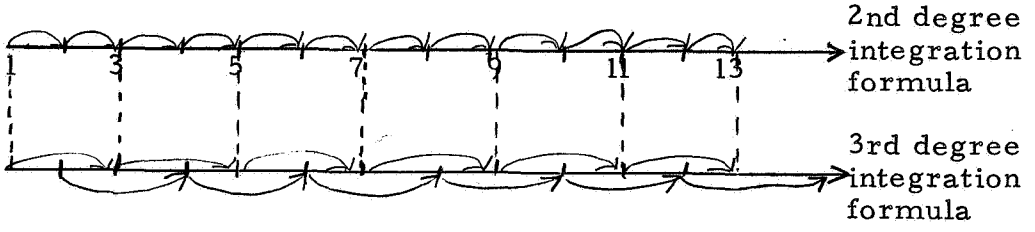
In the field of numerical analysis, a general ℓ^{th} degree integration formula can be represented by

$$x^*(i+\ell-1) = x^*(i) + \sum_{j=1}^{\ell} a_j z(i+j-1) \qquad (3\text{-}1)$$

where $x^*(\cdot)$ is the computed value of $x(\cdot)$ by applying the deterministic integration formula. The trapezoid, Simpson's and Newton's three-eighths rules are special cases of it. For example, $\ell = 2$ gives the trapezoid rule with $a_1 = a_2 = \frac{d}{2}$; $\ell = 3$ gives Simpson's rule with $a_1 = a_3 = \frac{d}{3}$ and $a_2 = \frac{4}{3}d$. In this section, a procedure is developed for calculating the cumulated mean square error generated by applying this ℓ^{th} degree integration formula. The criterion for judging the merit of these formulas is chosen to be the cumulated mean square error

$$J = \sum_{i=1}^{N} E[x(i) - x^*(i)]^2 \qquad (3\text{-}2)$$

Practically, the ℓ^{th} degree integration formula is applied at every $(\ell - 1)^{th}$ sampling point, that is, for $i = 1, \ell, 2\ell-1, 3\ell-2, \ldots$ The $x^*(i)$'s at other sampling points are not available. However, for the purpose of a fair comparison, additional $(\ell - 2)$ sets of ℓ^{th} degree integration formulas have to be carried out to cover the missed x(i)'s. For example, in the accompanying diagram



Two straight lines represent the time axes and the various marks on the lines indicate the places where the measurements are made. The 2nd degree integration formulas are carried out at $i = 1, 2, 3, 4, 5, \ldots$ (refer to the upper line), but for a 3rd degree integration formula (refer to the lower line) two sets of integration formulas have to be carried along, the one at $i = 1, 3, 5, \ldots$, the other at $i = 2, 4, 6, \ldots$, just for the purpose of computing J as shown in (3-2).

Considering the general ℓ^{th} degree integration formula (3-1) and substituting (2-8) into it, yields

$$x^*(i + \ell - 1) = x^*(i) + \sum_{j=1}^{\ell} a_j h^T y(i + j - 1) + \sum_{j=1}^{\ell} a_j v(i + j - 1) \qquad (3\text{-}3)$$

Recusively applying (2-7), gives

$$\begin{bmatrix} x(i + \ell - 1) \\ y(i + \ell - 1) \end{bmatrix} = \begin{bmatrix} \psi^{(\ell-1)} \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} + \sum_{k=1}^{\ell-1} \psi^{\ell-k-1} \begin{bmatrix} u_1(i + k - 1) \\ u_2(i + k - 1) \end{bmatrix}$$

$$+ \sum_{k=1}^{\ell-1} \psi^{\ell-1-k} \begin{bmatrix} w_1(i + k - 1) \\ w_2(i + k - 1) \end{bmatrix} \qquad (3\text{-}4)$$

Replacing y(i + j - 1) in (3-3) by the iteration equation in (3-4), we convert (3-3) into

$$x^*(i+\ell-1) = x^*(i) + \sum_{j=1}^{\ell-1} a_j h^T \Psi_{22}^{j-1} y(i) + \sum_{j=1}^{\ell} a_j h^T \sum_{k=1}^{j-1} \Psi_{22}^{j-k-1} [u_2(i+k-1)$$

$$+ w_2(i+k-1)] + \sum_{j=1}^{\ell} a_j v(i+j-1) \qquad (3\text{-}5)$$

Combining (3-5) and (3-4), and letting

$$X(i) = \begin{bmatrix} x^*(i) \\ x(i) \\ y(i) \end{bmatrix}$$

we have

$$X(i+\ell-1) = \Phi X(i) + U(i) + W(i) + S(i) \qquad (3\text{-}6)$$

where

$$\Phi = \left[\begin{array}{c|c} 1, \quad 0, & \sum_{j=1}^{\ell} a_j h^T \Psi_{22}^{j-1} \\ \hline 0, & \\ \vdots & \Psi^{(\ell-1)} \\ 0, & \end{array}\right]$$

$$U(i) = \begin{bmatrix} \sum_{j=1}^{\ell} a_j h^T \sum_{k=1}^{j-1} \Psi_{22}^{j-k-1} u_2(i+k-1) \\ \sum_{k=1}^{\ell-1} \Psi^{\ell-1-k} \begin{bmatrix} u_1(i+k-1) \\ \\ u_2(i+k-1) \end{bmatrix} \end{bmatrix}$$

$$W(i) = \begin{bmatrix} \sum_{j=1}^{\ell} a_j h \sum_{k=1}^{j-1} \Psi_{22}^{j-k-1} w_2(i+k-1) \\ \\ \sum_{k=1}^{\ell-1} \Psi^{\ell-1-k} \begin{bmatrix} w_1(i+k-1) \\ \\ w_2(i+k-1) \end{bmatrix} \end{bmatrix}$$

and

$$S(i) = \begin{bmatrix} \sum_{j=1}^{\ell} a_j v(i+j-1) \\ 0, \\ \vdots \\ 0, \end{bmatrix}$$

Examining the term S(i), we find that at the i^{th} step, it contains terms $a_1 v(i)$ through $a_\ell v(i+\ell-1)$, but at the next iteration, it contains $a_1 v(i+\ell-1)$ through $a_\ell v(i+2\ell-2)$. The presence of $a_\ell v(i+\ell-1)$ and $a_1 v(i+\ell-1)$ makes S(i) correlated with the next and proceeding iterations.

The mean equation of (3-6) is

$$\bar{X}(i+1) = \Phi \bar{X}(i) + u(i) \quad ; \quad \bar{x}(0) = x_o \tag{3-7}$$

where

$$\bar{X}(\cdot) \triangleq E[X(\cdot)]$$

Defining $$P(i) \triangleq E[X(i)X(i)^T] \tag{3-8}$$

from (3-6), yields

$$P(i+1) = \Phi P(i) \Phi^T + Q(i) + U(i)U(i)^T + \Phi \bar{X}(i) u(i)^T \tag{3-9}$$

$$= u(i)\bar{X}(i)^T \Phi^T + L \quad ; \quad P(0) = Po$$

where

$$Q(i) = E[w(i)w(i)^T]$$

and $$L = \begin{bmatrix} \sum_{j=1}^{\ell} a_j^2 R + 2a_1 a_\ell R & 0,\ldots,0 \\ 0,\ldots,0 & \\ 0 & 0,\ldots,0 \\ 0 & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & 0,\ldots,0 \end{bmatrix} ; \quad Po = \begin{bmatrix} x_o^2 & x_o^2 & 0,\ldots,0 \\ x_o^2 & x_o^2 & 0,\ldots,0 \\ 0, & 0, & \\ \cdot & \cdot & \\ \cdot & \cdot & p_o \\ 0 & 0 & \end{bmatrix}$$

The term, $2a_1 a_\ell R$, comes from the correlation effect of S(i). The criterion of (3-2) can now be equivalently expressed in terms of P as

$$J = \sum_{i=1}^{N} \text{Trace}\,[MP(i)]; \quad M = \begin{bmatrix} 1, & -1 & 0,\ldots,0 \\ -1, & 1, & 0,\ldots,0 \\ 0, & 0, & 0,\ldots,0 \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ 0, & 0, & 0,\ldots,0 \end{bmatrix}$$

(3-9) and (3-7) are two key equations. Applying (3-9) and (3-7) iteratively, we can obtain the error variances at all sampling points of the integration interval. The cost, J, of (3-10) is used as a base of comparison in this paper. Other cost functions, such as terminal mean square error, can serve the purpose too. An example, for the convenience of comparison with optimal integration, is given in the next section.

So far, we have treated the analysis problem on the known integration formulas, under random circumstances. In the next section, we deal with the problems of disigning on optimal integration formula which takes full advantage of the statistical knowledge of the integrand and which performs optimally in the sense of minimum mean square error.

In section 5, an optimal integration formula with complexity constraints will be discussed.

4. THEORETICAL APPROACH TO A MINIMUM VARIANCE DESIGN

In this section, an optimal integration formula for obtaining the best estimate of x(i) is developed by minimizing the cumulated mean square error.

Given process (2-7) and measurements (2-8), we require the optimal integration formula to be the one which takes the measurements, z(i), up to the present, into consideration, and in the meantime, minimizes the performance index

$$J = \sum_{i=1}^{N} E[x(i) - \hat{x}(i)]^2 \tag{4-1}$$

where $\hat{x}(i)$ is the best estimated value of x(i).

Since both (2-7) and (2-8) are linear, and we assume u(t) = 0, over the whole interval, it is reasonable to take the best estimate of x(i + 1), $\hat{x}(i+1)$, to be a linear combination of $\hat{x}(i)$, $\hat{y}(i)$ and the present measurement, z(i + 1), that is,

$$\begin{bmatrix} \hat{x}(i+1) \\ \\ \hat{y}(i+1) \end{bmatrix} = \begin{bmatrix} A(i) \end{bmatrix} \begin{bmatrix} \hat{x}(i) \\ \\ \hat{y}(i) \end{bmatrix} + \begin{bmatrix} B(i) \end{bmatrix} z(i+1) \tag{4-2}$$

where A(i) and B(i) are matrices and chosen to minimize J.

In order to solve this optimization problem, we employ the general technique[5] of converting a stochastic problem into a deterministic problem by working with means and variances.

Defining the variances

$$P(i) = E\left\{\begin{bmatrix} x(i) \\ \\ y(i) \end{bmatrix} \quad [x(i), y^T(i)]\right\}$$

$$\hat{P}(i) = E\left\{\begin{bmatrix} \hat{x}(i) \\ \\ \hat{y}(i) \end{bmatrix} \quad [\hat{x}(i), \hat{y}^T(i)]\right\}$$

and cross-correlation

$$S(i) = E\left\{\begin{bmatrix} x(i) \\ \\ y(i) \end{bmatrix} \quad [\hat{x}(i), \hat{y}^T(i)]\right\}$$

we can convert the performance index J into

$$J = \sum_{i=1}^{N} \text{Trace } D\,[P(i) + \hat{P}(i) - 2S(i)] \tag{4-3}$$

where

$$D = \begin{bmatrix} 1, 0, \ldots\ldots 0, 0 \\ 0, \quad\quad\quad\quad \cdot\ \cdot \\ \cdot \quad\quad\quad\quad \cdot\ \cdot \\ \cdot \quad\quad\quad\quad \cdot\ \cdot \\ \cdot \quad\quad\quad\quad \cdot\ \cdot \\ 0, 0, \ldots\ldots 0, 0 \end{bmatrix}$$

The equivalent minimization problem is to choose A(i) and B(i), $\forall i$, by minimizing (4-3). The solution of this problem is straightforward by applying the well-known technique of variational calculus (see appendix A) and the solution is

$$A(i) = \Psi(i) - B(i)H\Psi(i) \tag{4-4}$$

$$B(i) = M(i+1)H(HM(i+1)H^T + R)^{-1} \tag{4-5}$$

where

$$M(i+1) = P(i+1) - \Psi(i)S(i)^T\Psi(i)^T \tag{4-6}$$

$$H = [0, h^T]$$

The orthogonal condition, $E\{\hat{x}(i)[\hat{x}(i) - x(i)]\} = 0$, for all i, can be easily proved, that is to say, the linear estimator given in the form of (4-2) is the best linear estimator of x(i) (see appendix A). Furthermore, if all noises are gaussian distributed, (4-2), with A(i) and B(i) obtained from (4-4) and (4-5), $\hat{x}(i)$ is the conditional mean given all the measurements up to the present[6], or

$$\hat{x}(i) = E[x(i)/z(i), z(i-1), \ldots z(1)]$$

After a close examination of (4-2), (4-4), (4-5), and (4-6), we find a correspondence between this estimator and the well-known Kalman-Bucy filter, with B(i) given in (4-8), as the Kalman gain. However, in our derivation, no gaussian distributed noise is being assumed, whereas the gaussian distributed noise is a basic assumption in the Kalman-Bucy filter. Another noteworthy point of this derivation is that the same solution as (4-4), (4-5) and (4-6) is optimal for other performance indices J, such as the cumulated mean square error, the terminal mean square error, or even the mean square error on part of the state variables (see appendix A). A simple example is given below and a comparison between this optimal integration formula and the commonly used formulas (e. g. , trapezoid and Simpson's rules) is given by using the equations derived in section 3.

Example

INTEGRATION OF A FIRST-ORDER ZERO-MEAN RANDOM PROCESS

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \alpha y + w \\ z(i) &= y(i) \end{aligned} \tag{4-7}$$

where all variables are scalars and the sampling interval, d, is taken to be 1.

After discretization, we have

$$\Psi = \begin{bmatrix} 1, & \frac{e^{\alpha} - 1}{\alpha} \\ 0, & e^{\alpha} \end{bmatrix}$$

From (A-5), (A-6), (A-7) and (A-16), (A-17) in Appendix A we have

$$B(i) = \begin{bmatrix} -\frac{(1 - e^{\alpha})^2}{\alpha(1 - e^{2\alpha})} \\ 1 \end{bmatrix}$$

$$A(i) = \begin{bmatrix} 1, & \frac{e^{\alpha} - 1}{\alpha} + e^{\alpha}\left(-\frac{(1 - e^{\alpha})^2}{\alpha(1 - e^{2\alpha})}\right) \\ 0, & 1 \end{bmatrix}$$

Applying (4-2) and replacing $\hat{y}(i)$ by $z(i)$, we reach a very simple integration formula which carries the same computational load as the trapezoid rule.

$$\hat{x}(i + 1) = \hat{x}(i) + a_1 z(i) + a_2 z(i + 1)$$

where

$$a_1 = \left(\frac{e^{\alpha} - 1}{\alpha}\right) + e^{\alpha}\left(-\frac{(1 - e^{\alpha})^2}{\alpha(1 - e^{2\alpha})}\right)$$

$$a_2 = \frac{(1 - e^{\alpha})^2}{\alpha(1 - e^{2\alpha})} \qquad (4\text{-}8)$$

The weighting coefficients, a_1 and a_2, are functions of α only, and are independent of noise and time. Using (4-8), we calculate a_1 and a_2, for different values of α as shown in the following table:

α	a_1	a_2
0	0.5	0.5
-2	0.498	0.488
-5	0.2	0.2

It is interesting to note that the trapezoid rule is essentially the optimal integration formula for the case of $\alpha = 0$. For a comparison purpose, the numerical results obtained by applying the optimal integration formula are plotted together with those obtained by applying the trapezoid, Simpson's and rectangular rules, which are computed by using the equations derived in section 3. In figure 1, for $\alpha = 0$, where the integrand is comparatively smooth, the trapezoid rule can do as well as the optimal rule and Simpson's rule is the worst of all. For $\alpha = -18$, however, where the integrands are very random, the optimal integration formula takes advantage of the statistical information and as a result, a significant performance is obtained.

This example shows how the knowledge of statistical phenomena can help in providing better numerical results. If we try to extend this approach to random processes of higher order or more general random processes other than that of the exponentially correlated case, as in the example, we find the computational load for the equivalent Kalman-Bucy filter become larger and larger, (i.e. Eqs (4-2), (4-4) and (4-5) since for a higher order random process, more equations have to be carried along to update the best estimate of x('). This, in general, is not satisfactory. Hence, in the next section, we would like to consider a procedure of developing an optimal integration formula with computational constraints.

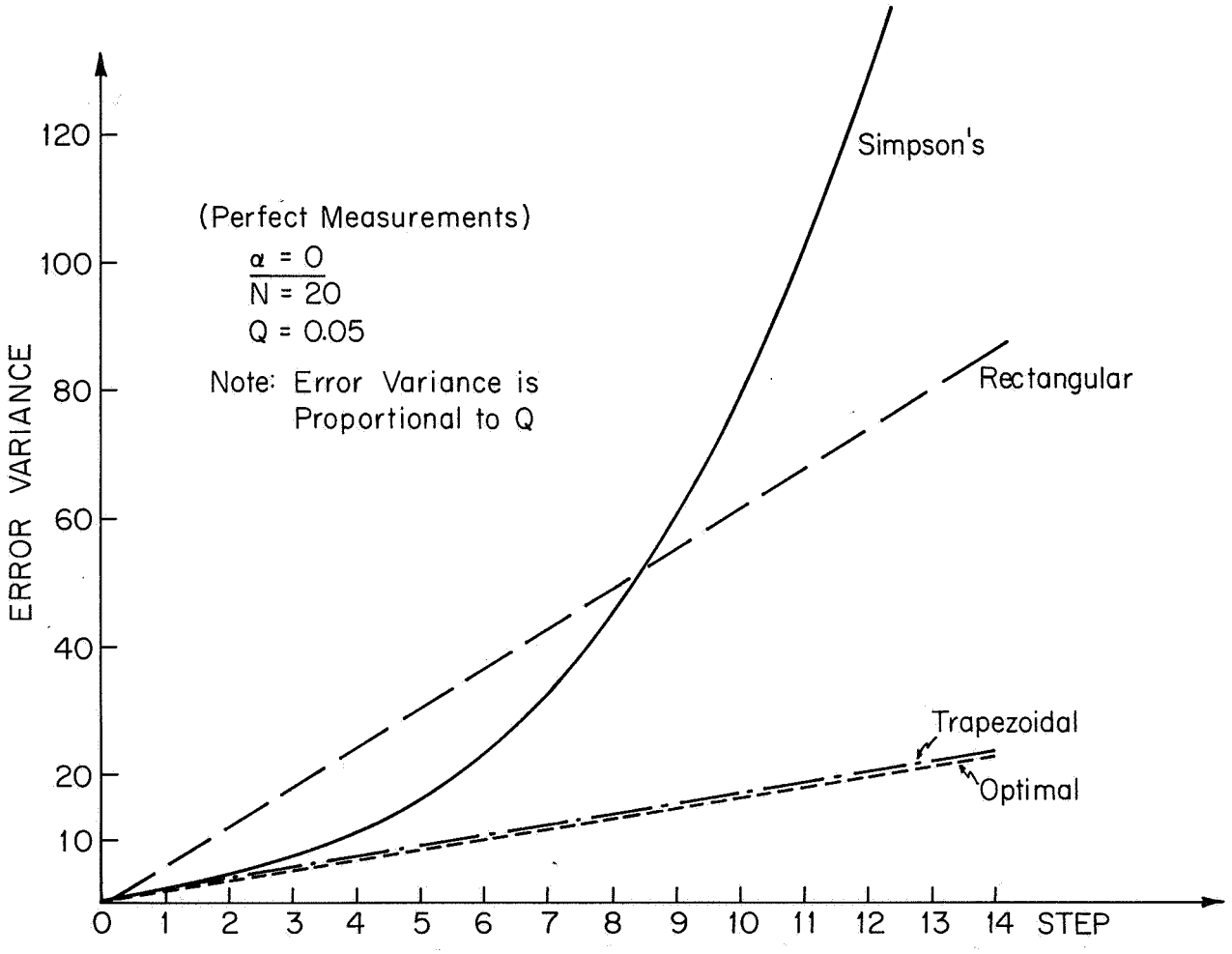


FIG. 1 STATISTICAL EVALUATION OF INTEGRATION FORMULAS

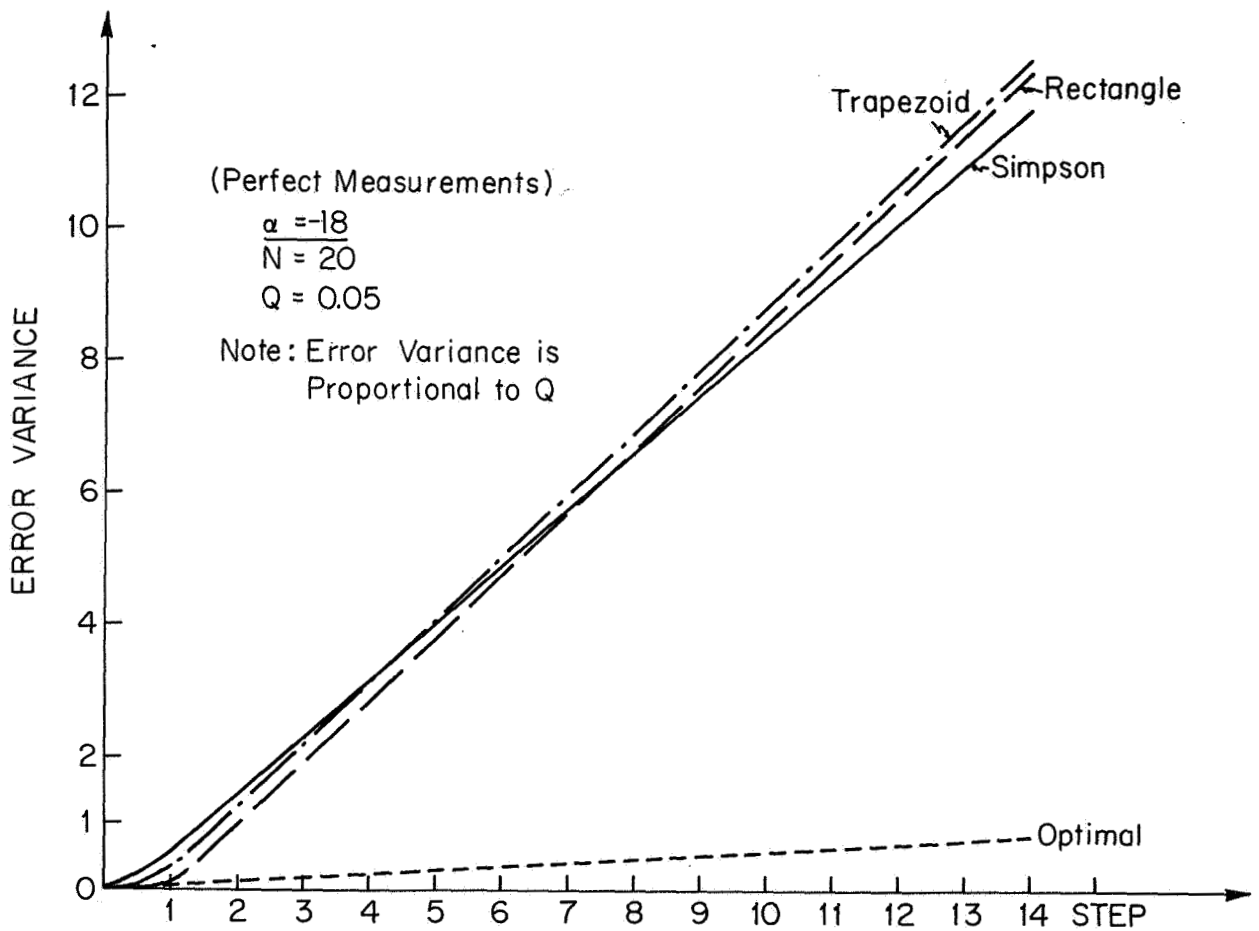


FIG. 2 STATISTICAL EVALUATION OF INTEGRATION FORMULAS

5. OPTIMAL INTEGRATION FORMULAS SUBJECT TO COMPUTATIONAL CONSTRAINTS

The computational constraints considered in this section are to require the optimal integration formula carrying the same computational load as that of the deterministic integration formula no matter how high the order of the random process is.

A class of integration formulas will be developed, which is given as follows

$$\hat{x}(i + \ell - 1) = \hat{x}(i) + \sum_{j=1}^{\ell} a_j(i) z(i + j - 1) \qquad (5\text{-}1)$$

where the new estimate of the integral is given by the old value of the integral at previous times plus a linear combination of past measurements, and where ℓ is the degree of the integration formula in conformity with the usual terminology of numerical analysis. The $a_j(i)$'s are to be determined to minimize the performance index which is the same one as given in section 3 and 4.

$$J = \sum_{i=1}^{N} E[x(i) - \hat{x}(i)]^2$$

For $\ell = 2$ the optimal integration formula will have the form of the trapezoid rule except that the weighting coefficients are to be determined. That is

$$\hat{x}(i + 1) = \hat{x}(i) + a_1(i) z(i) + a_2(i) z(i + 1) \qquad (5\text{-}2)$$

In this section, only the 2nd degree optimal integration formula will be considered.

Following the same steps in section 3, and defining

$$X(i) \triangleq \begin{bmatrix} \hat{x}(i) \\ x(i) \\ y(i) \end{bmatrix}$$

$$P(i) \triangleq E[X(i)X(i)^T]$$

we reach the same iterative relations for P(i) as in (3-9), for $\ell = 2$, except for replacing all a_1 by $a_1(i)$ and a_2 by $a_2(i)$.

That is

$$P(i+1) = \Phi(i)P(i)\Phi(i)^T + Q(i) + U(i)U(i)^T + \Phi(i)\bar{X}(i)U(i)^T + U(i)\bar{X}(i)^T\Phi(i)^T + L(i) \qquad (5\text{-}3)$$

$$; \; P(o) = Po$$

where

$$L(i) = \begin{bmatrix} a_1(i)^2R + a_2(i)^2R + 2a_1(i)a_2(i)R, & 0, \ldots\ldots, 0 \\ 0, & 0, \ldots\ldots, 0 \\ \vdots & \vdots \\ 0, & 0, \ldots\ldots, 0 \end{bmatrix}$$

J is converted into

$$J = \sum_{i=1}^{N} MP(i) \; ; \qquad M = \begin{bmatrix} 1, & -1, & 0, \ldots, 0 \\ -1, & 1, & 0, \ldots, 0 \\ 0, & 0, & \cdots \\ \vdots & \vdots & \vdots \\ 0, & 0, & 0, \quad 0 \end{bmatrix} \qquad (5\text{-}4)$$

Now, the equivalent optimization problem is to choose $a_1(i)$ and $a_2(i)$ for all i to minimize (5-4), subject to the constraint (5-3). This is a standard deterministic optimization problem except for the presence of the delayed control $a_2(i-1)$ in L(i). This difficulty can be easily overcome by defining another control variable $T(i+1) = a_2(i)$ and viewing this equivalent relation as the control variable constraints. By the general variational technique, we define a Hamiltonian

$$\begin{aligned}\mathcal{H}(i) = \text{Trace} \Big\{ & MP(i) + \Lambda(i+1)[\Phi(i)P(i)\Phi(i)^T + Q(i) \\ & + U(i)U(i)^T + \Phi(i)\bar{X}(i)U(i)^T + U(i)\bar{X}(i)\Phi(i)^T \\ & + L(i)] + \eta(i+1)a_2(i)] \Big\}\end{aligned} \tag{5-5}$$

where $\Lambda(i+1)$ and $\eta(i+1)$ are lagrange multipliers of the constraints (5-3) and $T(i+1) = a_2(i)$, respectively.

$$\text{Setting} \qquad \frac{\partial \mathcal{H}(i)}{\partial P(i)} = \Lambda(i); \quad \frac{\partial \mathcal{H}(i)}{\partial T(i)} = \eta(i) \tag{5-6}$$

we have the variation on J

$$\delta J = \sum_{i=1}^{N-1} \left(\frac{\partial \mathcal{H}(i)}{\partial a_1(i)} \delta a_1(i) + \frac{\partial \mathcal{H}(i)}{\partial a_2(i)} \delta a_2(i) \right) \tag{5-7}$$

The necessary conditions for $a_1(i)$ and $a_2(i)$ to be optimal are

$$\begin{aligned} \frac{\partial \mathcal{H}(i)}{\partial a_1(i)} &= 0 \\ & \qquad \forall_i \\ \frac{\partial \mathcal{H}(i)}{\partial a_2(i)} &= 0 \end{aligned} \tag{5-8}$$

Eqs. (5-3), (5-6) and (5-8) constitute a two point boundary value problem. An analytical solution to this problem, in general, is either hard to reach or unavailable. The numerical minimization technique is applied. In this paper we adopt the well-known steepest descent gredient method and a numerical example is solved by applying it.

In general, the weighting coefficients $a_1(i)$ and $a_2(i)$ are time-varying which may be inconvenient in real time applications. To eliminate this difficulty, a procedure for designing a piece-wise constant coefficient formula is developed.

In the case of requiring $a_1(i)$ and $a_2(i)$ to be constant over the whole integration interval, we follow the same approach for obtaining time-varying optimal $a_1(i)$ and $a_2(i)$ as we did up to (5-7) and then write (5-7) as

$$\delta J = \left[\sum_{i=1}^{N-1} \frac{\partial \mathcal{H}(i)}{\partial a_1}\right] \delta a_1 + \left[\sum_{i=1}^{N-1} \frac{\partial \mathcal{H}(i)}{\partial 2_a}\right] \delta a_2$$

For small variations in δa_1 and δa_2, and requiring $\delta J = 0$, we have

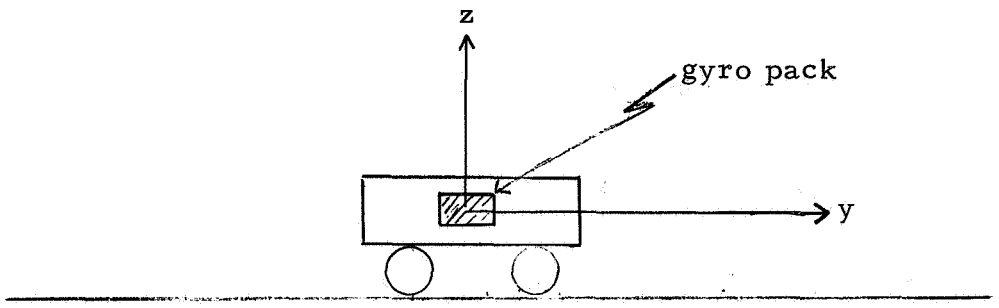
$$\sum_{i=1}^{N-1} \frac{\partial \mathcal{H}(i)}{\partial a_1} = 0$$

$$\sum_{i=1}^{N-1} \frac{\partial \mathcal{H}(i)}{\partial a_2} = 0 \qquad (5\text{-}9)$$

as contrasted with (5-8). The solution essentially remains the same. Eq. (5-9) can be very easily extended to the case of piece-wise constant coefficients. The restraints on constant or piece-wise constant $a_1(i)$ and $a_2(i)$ would of course result in a decrease in accuracy.

EXAMPLE: INTEGRATING THE GYRO OUTPUT

In a gyro performance test,[3] a gyro pack is installed in a sled which is tested on a long horizontal track. Gyros are used to measure the angular velocity of the sled about the body axes y, z and x, where x is perpendicular to the paper



From a consideration of the characteristics of gyro and vehicle dynamics, v_x, the angular velocity of the sled about the x-axis, can be modelled as a 5th order gaussian Markov random process driven by white noise and v_z, as a 3rd order gaussian Markov random process.[3] v_x and v_z are uncorrelated. We are interested in the angular derivations of the gyro pack from the x-axis and the z-axis which normally would be zero. The differential equation, which describes the v_x process, is shown as follows

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \end{bmatrix} = \begin{bmatrix} 0, & B\omega_{y_2}^2, & \frac{B\omega_{y_2}}{Q^1_1}, & A\omega_{y_1}^2, & \frac{A\omega_{y_1}}{Q^1_0} \\ 0, & 0, & 0, & 0, & 0 \\ 0, & -\omega_{y_2}^2, & \frac{-\omega y_2}{Q^1}, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & -\omega_{y_1}^2, & \frac{-\omega y_1}{Q^1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

where we take y_1 as v_x and $\omega_{y_1}, \omega_{y_2}$ are two natural frequencies of the vibration of the vehicle, and

$$Q^1 = 5$$

$$\omega_{y_1} = 60\pi \text{ rad/sec.}$$

$$\omega_{y_2} = 320\pi \text{ rad/sec.}$$

$$A = 0.195 \text{ rad/sec.}^2$$

$$B = 0.0399 \text{ rad/sec.}^2$$

The gyro dynamics are neglected here. The measurements are taken from gyro outputs which are computed by the white noise

$$z(i) = y_1(i) + v(i)$$

where

$$E[v(i)] = 0$$

$$E[v(i)v(j)] = R\delta ij; \quad R = 0.01$$

We are interested in obtaining the angular deviation that is, the integrated value of the angular velocity of the vehicle, or $\dot{x} = y_1; x(0) = 0$. The numerical results for zero, parabolic and sinusoidal input (i.e. u(t)) are given below over 30 integration steps. Improvement by applying optimal integration rules is clearly shown, especially in the zero-mean ease. 25% improvement over that of the trapezoid rule is expected, if we apply it to the x-axis. Even greater improvement is expected, if we apply it to the z-axis. The cost for deterministic integration formulas is computed by applying the results in section 3. In the case of u(t) = 2t, or equivalently t^2 input to the integrator x(t), the costs for Simpson's rule and Newton's three-eighths rule remain unchanged as do those in the zero mean case, since Simpson's rule and Newton's three-eighths rule can integrate parabolic functions perfectly; the deterministic error is zero. The only error is due to random error.

The cost for applying an optimal integration formula to the z-axis is 0.332×10^{-8} rad^2 which shows an improvement; and that for applying the trapezoid rule is 0.98×10^{-7} rad^2. The optimal values of $a_1(i)$ and $a_2(i)$ for this case are shown in figure 3.

In the case that $a_1(i)$ and $a_2(i)$ are constrained to be constant over the whole integration interval, then, for the x-axis case, we have

$$J = 0.413 \times 10^{-5}$$

$$a_1 = 0.4896 \times 10^{-3}$$

$$a_2 = 0.4893 \times 10^{-3}$$

Only 4% improvement is expected.

For the z-axis case, we have

$$J = 0.1583 \times 10^{-6}$$

$$a_1 = 0.1808 \times 10^{-3}$$

$$a_2 = 0.1877 \times 10^{-3}$$

25% improvement is expected.

J of u(t)	Optimal Formula	Trapezoidal	Simpson's	Newton's 3/8
0	0.30×10^{-5}	0.4205×10^{-5}	0.616708×10^{-5}	0.438118×10^{-5}
2t		0.6677×10^{-4}	0.616708×10^{-5}	0.438118×10^{-5}
Sin 483t		0.1410×10^{-4}	0.616708×10^{-5}	0.438118×10^{-5}

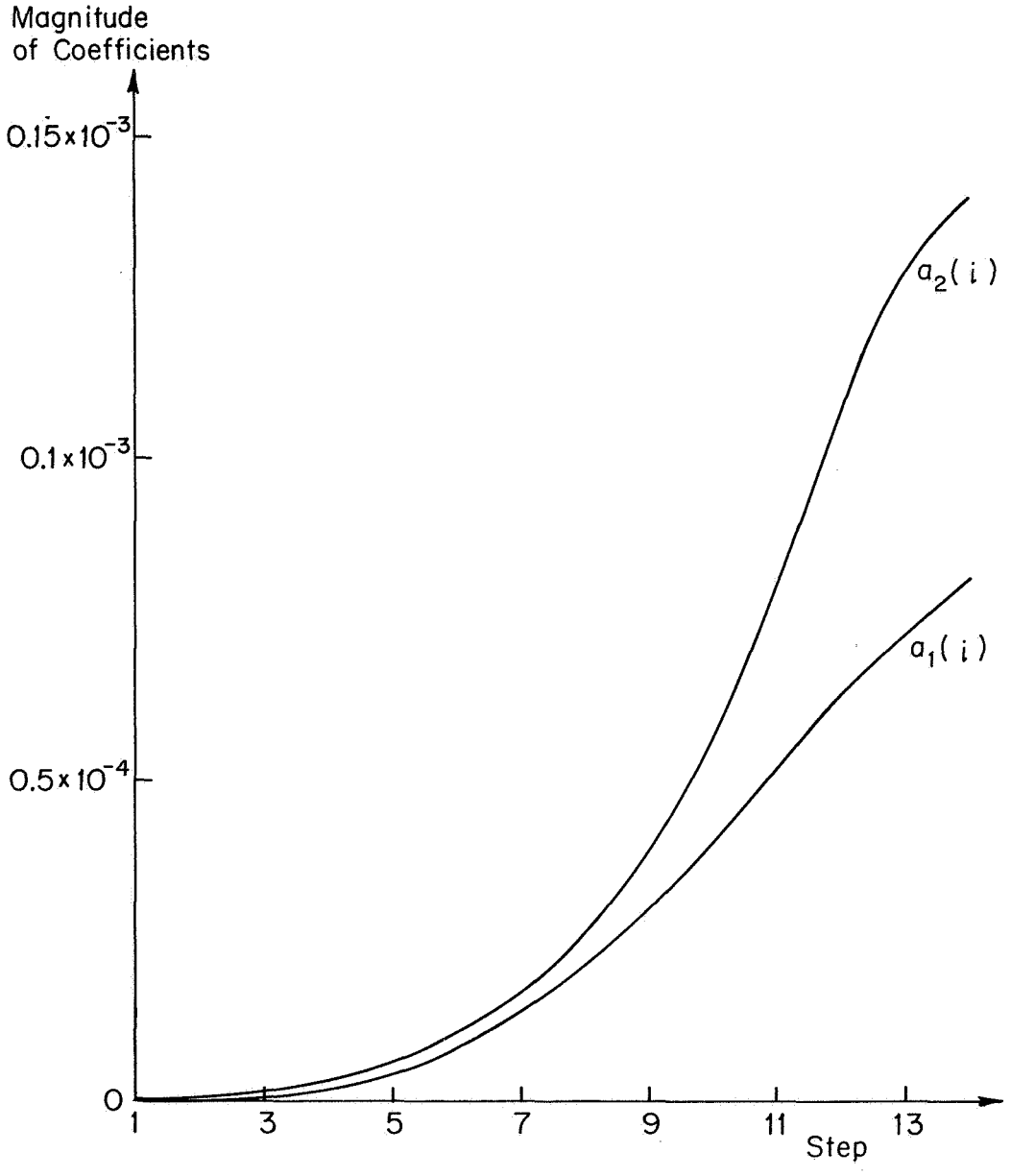
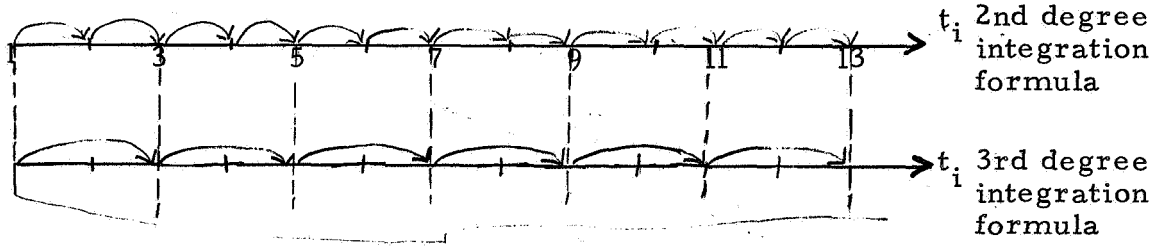

Magnitude
of Coefficients
0.15x10^-3
0.1x10^-3
0.5x10^-4
0
1
3
5
7
9
11
13
Step
a2(i)
a1(i)

FIGURE 3

6. A HIGHER DEGREE INTEGRATION FORMULA

A higher degree integration formula generally requires less computation to cover the same integration interval, but it is used at the expense of output frequency of the integrated value (refer to Figure 4). In principle, given a certain amount of computational capacity, it is possible to compute the optimal integration formula of appropriate degree to fit it. However, it turns out that we need not to repeat the computations for the formulas of different degrees. We only have to get the 2nd degree integration formula and then all higher degree formulas can be derived directly from it



Common Output Points

As an illustration (see the accompaying diagram), for a 2nd degree formula, x(•) is available at every sampling point, but for a 3nd degree formula, it is available ony at every other sampling point. Thus, if only accuracies at common output points are important, the coefficient $a_j^t(i)$ for the 3rd degree formula can be directly obtained from $a_j(i)$ for the 2nd degree formula, by the following relations

$$\begin{aligned} a_1^t(i) &= a_1(i) \\ a_2^t(i) &= a_2(i) + a_1(i+1) \\ a_3^t(i) &= a_2(i+1) \end{aligned} \qquad (6\text{-}1)$$

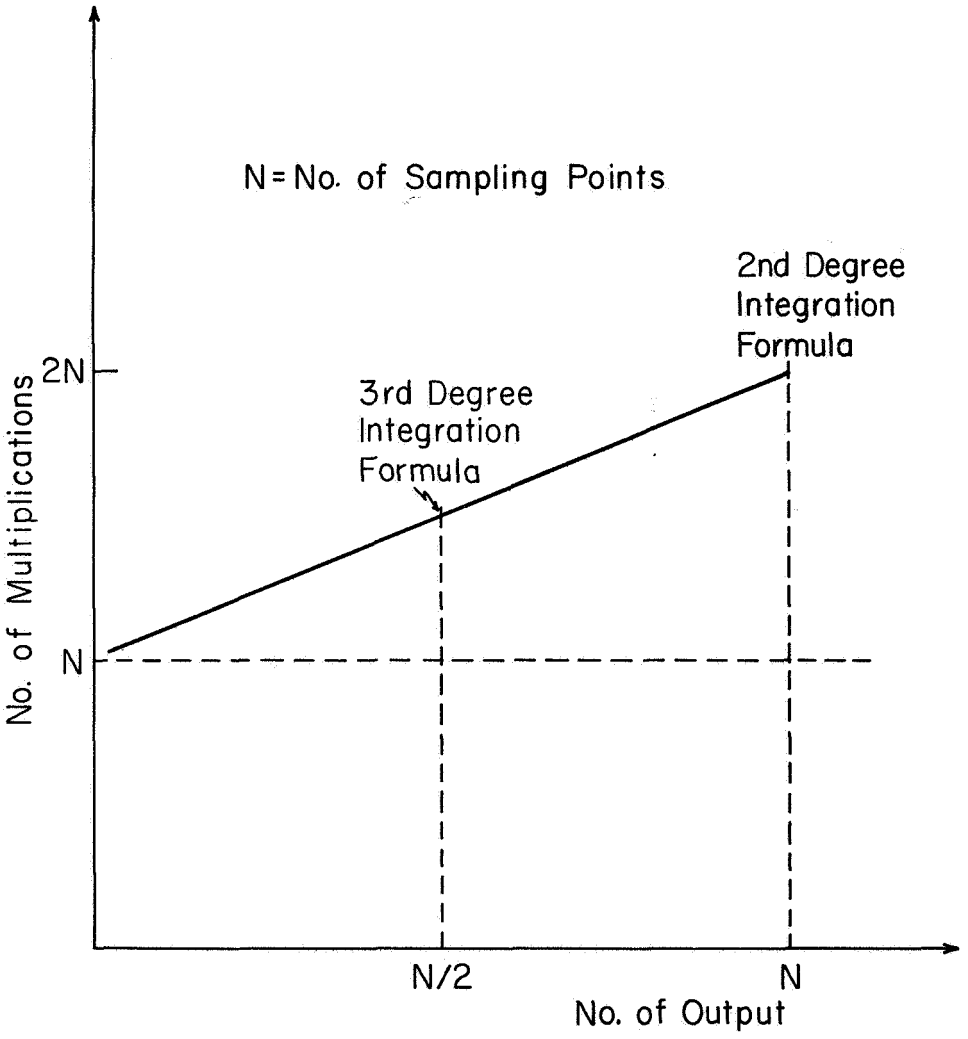

N = No. of Sampling Points
2nd Degree
Integration
Formula
3rd Degree
Integration
Formula
2N
N
No. of Multiplications
N/2
N
No. of Output

FIGURE 4

These relations can be generalized to higher degree. This means that once we have obtained the coefficients for the 2nd degree formula, the coefficients for other degree formulas can be directly obtained without having to solve another optimization problem. This assertion is proved in Appendix B, for interested readers.

Since the accuracies of optimal formulas of different degrees at common output points are the same, a trade-off between computational speed and output frequency is possible. As far as computational speed is concerned, only the number of multiplications will be counted, since it is the main contributor to computing. Let N, a very large number, be the number of sampling points. Then the number of multiplications required to be carried out over this integration period by using the ℓ^{th} degree formula is $\frac{\ell}{\ell-1}N$. The number of output points given by applying the ℓ^{th} degree integration formula is $N/(\ell-1)$. The plot of the number of multiplications against the number of outputs is shown in Figure 4. It is a linear curve. For instance, for 2nd degree integration formulas, the number of multiplications required is 2 N and the number of outputs is N. But as the degree goes higher, the computational load is decreased, and so is the number of outputs. Thus, if one is willing to accept less outputs but at equivalent accuracy, he can get away with lower computational loads. Therefore, depending on the speed of an onboard computer, an integration formula of appropriate degree which will achieve equal accuracy at common output points, can be chosen by using Figure 4. This kind of trade-off cannot be done for deterministic integration rules, since the accuracies of the integration at the common output points are not equal.

7. CONCLUSION

It has been shown in this report how to design an optimal integration formula by taking the statistical knowledge of the integrand into account, and also how to derive higher degree formulas from a 2nd degree formula. A trade-off between output frequency and computational load has been discussed. In any case, the authors strongly believe that this work would prove useful in space navigation applications.

Further researches and developments in this area are expected. An interesting and important problem in this area is to find the updating numerical formula for a directional cosine differential equation which is used in the strapdown navigation system for updating the attitude of a space vehicle.

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APPENDIX A

MINIMUM VARIANCE FILTER

In this appendix, instead of considering the problem formulated in section 4, we give a general derivation of the minimum variance filter. The optimal integration formula shown in section 4 will be a special case of this.

Consider a Markov random sequence

$$x(i+1) = \Phi(i)x(i) + w(i);\ E[x(o)] = x_o,\ E[x(o)x(o)^T] = Po;\ i = 0, \ldots, N \tag{A-1}$$

where $x(\cdot)$ is an $n \times 1$ vector, $\Phi(\cdot)$ is an $n \times n$ transition matrix, and $w(\cdot)$ is the n-dimensional white noise with

$$E[w(i)] = 0$$

$$E[w(i)w(j)^T] = Q\delta_{ij}$$

The measurement we have is

$$z(i) = H(i)x(i) + v(i) \tag{A-2}$$

where $z(i)$ is an $\ell \times 1$ vector, $H(i)$ is an $\ell \times n$ matrix and $v(i)$ is an ℓ-dimensional white random noise with

$$E[v(i)] = 0$$

$$E[v(i)v(j)^T] = R\delta_{ij}$$

Since (A-1) and (A-2) are linear, we assume the optimal estimate of $x(i+1)$, $\hat{x}(i+1)$, to be a linear combination of the previous best estimate of $x(i)$, $\hat{x}(i)$, and the present measurement $z(i+1)$. That is,

$$\hat{x}(i+1) = A(i)\hat{x}(i) + B(i)z(i+1) \tag{A-3}$$

where $A(i)$ is an $n \times n$ matrix and $B(i)$ is an $n \times \ell$ matrix, and $A(i)$ and $B(i)$ are to be determined so as to minimized the performance index J, which is,

$$J = \sum_{i=0}^{N} E \| x(i) - \hat{x}(i) \|^2_{D(i)} \tag{A-4}$$

The presence of D(i) increases the generality of J. For example, puting D(i) = 0 for i = 0, 1, . . . , (N - 1), would result in a terminal variance minimization problem. D(i) is assumed to be symmetrical.

Defining the variance

$$P(i) = E[x(i)x(i)^T]$$
$$\hat{P}(i) = E[\hat{x}(i)\hat{x}(i)^T]$$

and cross correlation

$$S(i) = E\{x(i)\hat{x}(i)^T\}$$

we derive the following iterative equations for P(i), $\hat{P}(i)$, and S(i) by using (A-1), (A-2) and (A-3),

$$P(i+1) = \Phi(i)P(i)\Phi(i)^T + Q(i) \quad ; \quad P(o) = Po \tag{A-5}$$

$$\hat{P}(i+1) = A(i)\hat{P}(i)A(i)^T + B(i)H(i+1)P(i)H(i+1)^T B(i)^T + B(i)RB(i)^T + B(i)H(i+1)\Phi(i)S(i)A(i)^T + A(i)S(i)^T\Phi(i)^T H(i+1)^T B(i)^T \quad ; \quad \hat{P}(o) = Po \tag{A-6}$$

$$S(i+1) = \Phi(i)S(i)A(i)^T + P(i+1)H(i+1)^T B(i)^T \; ; \; S(o) = Po \tag{A-7}$$

and the performance index (4) is converted into

$$J = \text{Trace}\left\{\sum_{i=0}^{N} D(i)[P(i) + \hat{P}(i) - S(i) - S(i)^T]\right\} . \tag{A-7}$$

Since D(i) is a symmetrical matrix

$$J = \text{Trace}\left\{\sum_{i=0}^{N} D(i)[P(i) + \hat{P}(i) - 2S(i)]\right\} . \tag{A-9}$$

Now, the equivalent optimization problem is to choose A(i) and B(i) to minimize (A-9), subject to (A-5), (A-6) and (A-7) as constraints. Following the well-known technique of variational calculus, define the Hamiltonian,

$$\mathcal{H}(i) = \text{Trace}\left\{D(i)[P(i) + \hat{P}(i) - 2S(i)] + \xi(i+1)\hat{P}(i+1) + \eta(i+1)S(i+1)\right\} \tag{A-10}$$

where $\xi(\cdot)$ and $\eta(\cdot)$ are Lagrange multipliers with dimension $n \times n$. (A-5) is not jointed to (A-10), since (A-5) is not a function of the control variables.

Necessary conditions are listed as follows:

From $\frac{\partial \mathcal{H}(i)}{\partial \hat{P}(i)} = \xi(i)$ and $\frac{\partial \mathcal{H}(i)}{\partial S(i)} = \eta(i)$ we have

$$\xi(i) = D(i) + A(i)^T \xi(i+1) A(i) \quad ; \quad \xi(N) = D(N) \tag{A-11}$$

$$\eta(i) = 2A(i)^T \xi(i+1)\, B(i)H(i+1)\Phi(i) + A(i)^T \eta(i+1)\Phi(i) \tag{A-12}$$
$$-2D(i) \quad ; \quad \eta(N) = -2D(N)$$

From $\frac{\partial \mathcal{H}(i)}{\partial A(i)} = 0$ and $\frac{\partial \mathcal{H}(i)}{\partial B(i)} = 0$, yields

$$2\xi(i+1)[A(i)\hat{P}(i) + B(i)H(i+1)\Phi(i)S(i)] + \eta(i+1)[\Phi(i)S(i)] = 0 \tag{A-13}$$

$$2\xi(i+1)[B(i)H(i+1)P(i)H(i+1)^T + B(i)R + A(i)S(i)^T\Phi(i)^T H(i+1)^T]$$
$$+ \eta(i+1)[P(i+1)H(i+1)^T] = 0 \tag{A-14}$$

Eqs. (A-6, A-7), (A-11, A-12), (A-13, A-14) constitute a two point boundary-value problem. In general, the analytical solution is not available for a two-point boundary-value problem. In our case, however, this is possible. It is easily seen from (A-13) and (A-14). If we let

$$\eta(i) = -2\xi(i) \text{ for } i = 0, \ldots, N \tag{A-15}$$

then

$$A(i)\hat{P}(i) = [\Phi(i) - B(i)H(i+1)\Phi(i)]S(i) \tag{A-16}$$

$$B(i)H(i+1)P(i)H(i+1)^T + B(i)R + A(i)S(i)^T\Phi(i)^T H(i+1)^T$$
$$= P(i+1)H(i+1)^T \tag{A-17}$$

We will verify the truth of (A-15) later.

From the initial condition $\hat{P}(o) = S(o)$, (A-16) becomes

$$A(o) = \Phi(o) - B(o)H(1)\Phi(o). \tag{A-18}$$

Substituting (A-19) into (A-17), we have

$$B(o) = [P(1)H(1)^T - \Phi(o)S(o)^T\Phi(o)^TH(1)^T][H(1)P(1)H(1)^T - H(1)\Phi(o)S(o)^T\Phi(o)^TH(1)^T + R]^{-1}. \tag{A-19}$$

If $\hat{P}(1) = S(1)$, then we can carry (A-18) and (A-19) a step further and iteratively, we can prove (A-18) and (A-19) are true and $\hat{P}(i) = S(i)$, for all i. That is,

$$A(i) = \Phi(i) - B(i)H(i+1)\Phi(i) \tag{A-20}$$

$$B(i) = [P(i+1)H(i+1)^T - \Phi(i)S(i)^T\Phi(i)^TH(i+1)^T]. \tag{A-21}$$

$$[H(i+1)P(i+1)H(i+1)^T - H(i+1)\Phi(i)S(i)^T\Phi(i)^TH(i+1)^T + R]^{-1}$$

To prove $\hat{P}(1) = S(1)$, we substitute (A-18) and (A-19) into (A-6). After manipulation, this yields

$$\hat{P}(1) = \Phi(o)\hat{P}(o)\Phi(o)^T + [P(1)H(1)^T - \Phi(o)S(o)\Phi(o)^TH(1)^T]B(o)^T. \tag{A-22}$$

Similarly, substituting (A-18) and (A-19) into (A-7), we have

$$S(1) = \Phi(o)S(o)\Phi(o)^T + [P(1)H(1)^T - \Phi(o)S(o)\Phi(o)^TH(1)^T]B(o)^T \tag{A-23}$$

(A-22) and (A-23) imply $\hat{P}(1) = S(1)$ and by induction, we prove

$$\hat{P}(i) = S(i) \tag{A-24}$$

The Orthogonality Principle:

The interpretion of (A-24) leads to the orthogonality principle of minimum mean square estimation. This principle states that $\hat{x}(i)$ is the optimal linear estimate of x(i), for all i, if and only if the error vector $e(i) \triangleq x(i) - \hat{x}(i)$ is orthogonal to $\hat{x}(i)$, that is, $E[e(i)\hat{x}(i)] = 0$, for all i, or $S(i) = \hat{P}(i)$. Thus, if (A-15) is true the filter derived so far is the optimal linear filter in the minimum mean square sense.

Now, we go back to prove (A-15), which says,

$$\eta(i) = -2\xi(i) \quad ; \quad \forall_i. \tag{A-25}$$

Rewriting (A-20) as

$$B(i)H(i+1)\Phi(i) = \Phi(i) - A(i)$$

and substituting it into (A-12), we have

$$\eta(i) = A(i)^T[2\xi(i+1) + \eta(i+1)]\Phi(i) - 2A(i)^T\xi(i+1)\,A(i) - 2D(i). \tag{A-26}$$

Multiplying (A-11) by 2 and adding to (A-26), yields

$$[\eta(i) + 2\xi(i)] = A(i)^T[2\xi(i+1) + \eta(i+1)]\Phi(i) \tag{A-27}$$

From the boundary conditions $\xi(N) = D(N)$ and $\eta(N) = -2D(N)$, we have

$$\eta(N) + 2\xi(N) = 0$$

Applying (A-27) iteratively, we prove that

$$\eta(i) + 2\xi(i) = 0 \quad ; \quad \forall_i$$

Thus far, we have derived the optimal filter having the form (A-3) with A(i) and B(i) given by (A-20) and (A-21), which can be calculated by using (A-5), (A-6) and (A-7). Furthermore from the orthogonality principle, we proved the filter given by (A-3) to be the optimal linear mean square filter. Two points to be emphasized here are that first, no gaussian assumption on the noise probability density function is made and second, in the computing process to obtain A(i) and B(i), D(i) does not appear in the calculations; that is, the solution given by (A-3), (A-21) and (A-20) remains the same no matter what D(i) is. This justifies the statement in section 4 that the integration formula is optimal for the performance index other than J given by Eq. (4-1).

The connection of this minimum variance filter with the well-known Kalman-Bucy filter is that B(i) given by (A-21), is exactly the Kalman gain at i + 1. This filter can be viewed as an alternative derivation of the Kalman-Bucy filter.

SUMMARY OF RESULTS

Plant

$$x(i+1) = \Phi(i)x(i) + w(i) \; ; \; E[x(o)x(o)^T] = Po$$

Measurements

$$z(i) = H(i)x(i) + v(i) \; ; \; E[v(i)] = 0, \; E[v(i)v(j)^T] = R\delta_{ij}$$

Minimum variance Filter

$$\hat{x}(i+1) = A(i)\hat{x}(i) + B(i)z(i+1)$$

Performance Index

$$J = \sum_{i=o}^{N} E \| \hat{x}(i) - x(i) \|_{D(i)}$$

A(i) and B(i) are given by

$$A(i) = \Phi(i) - B(i)H(i+1)\Phi(i)$$

$$B(i) = [P(i+1)H(i+1)^T - \Phi(i)S(i)^T\Phi(i)^T H(i+1)^T] \cdot$$

$$[H(i+1)P(i+1)H(i+1)^T - H(i+1)\Phi(i)S(i)^T\Phi(i)^T H(i+1)^T + R]^{-1}$$

where P(i) and S(i) are calculated by

$$P(i+1) = \Phi(i)P(i)\Phi(i)^T + Q(i) \quad ; \quad P(o) = Po$$

$$S(i+1) = \Phi(i)S(i)A(i)^T + P(i+1)H(i+1)^T B(i)^T \quad ; \quad S(o) = Po$$

APPENDIX B

DERIVATION OF A HIGHER DEGREE INTEGRATION FORMULA FROM THE 2ND DEGREE FORMULA.

Only the derivation and proof for the case of obtaining 3rd degree weighting coefficients from 2nd degree coefficients will be given. Other higher degree formulas will be a direct extention.

Two optimization problems.

(1) 2nd degree integration formula

$$\hat{x}(i+1) = \hat{x}(i) + a_1(1)z(i) + a_2(i)z(i+1); \quad \forall_i \tag{B-1}$$

$a_1(i)$ and $a_2(i)$ are optimally chosen such that

$$J = \sum_{i=1}^{N/2} MP(2i) \text{ is a minimum, subject to constraint (5-3)}$$

which is

$$P(i+1) = \Phi(i)P(i)\Phi(i)^T + Q(i) + \ldots\ldots + L(i) \tag{B-2}$$

(2) 3rd degree intergration formula

$$x(i+2) = x(i) + a_1^t(i)z(i) + a_2^t(i)z(i+1) + a_3^t(i)z(i+2) \tag{B-3}$$

$$\text{for } i = 1, 3, 5, 7, \ldots, N$$

$a_1^t(i), a_2^t(i)$ and $a_3^t(i)$ are optimally chosen such that

$$J = \sum_{i=1}^{N/2} MP(2i) \text{ is a minimum, subject to the constraint of}$$

similar form as (B-2).

Now we want to prove that these two minimization problems will give the same optimal cost J by applying the following relations between these two sets of coefficients.

$$a_1^t(i) = a_1(i)$$

$$a_2^t(i) = a_2(i) + a_1(i+1) \qquad \text{for } i = 1, 3, 7, \ldots, N \qquad (B-4)$$

$$a_3^t(i) = a_2(i+1)$$

Proof: If we let

$$a_2^t(i) = a_{21}^t(i) + a_{22}^t(i)$$

then (B-3) can be written as

$$\hat{x}(i+1) = \hat{x}(i) + a_1^t(i)z(i) + a_{21}^t(i)z(i+1)$$

$$\hat{x}(i+2) = \hat{x}(i+1) + a_{22}^t(i)z(i+1) + a_3^t(i)z(i+2).$$

The equation between P(i + 2) and P(i) for the 3rd degree formula can also be written as two sets of (B-2).

If we let

$$a_1^t(i) = a_1(i)$$

$$a_{21}^t(i) = a_2(i) \qquad \text{for } i = 1, 3, 7, \ldots, N$$

$$a_{22}^t(i) = a_1(i+1)$$

$$a_3^t(i) = a_2(i+1)$$

and if $a_1^t(i), a_{21}^t(i), a_{22}^t(i)$ and $a_3^t(i)$ are optimally chosen, $a_2^t(i)$ will be optimally determined by the relation (B-5). Thus we have proved that the optimization problem (B-2) is equivalent to the optimization problem (B-1) by the relation (B-4).